JAWAHARLAL NEHRU COLLEGE BOKO

DEPARTMENT OF MATHEMATICS

A Project submitted for the partial fulfillment of the studies of B.Sc. (Hons.) 6th semester.

Topic

An Application of Fourier Analysis to understand Musical Sounds and Principle of Additive Synthesis.

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"This project is dedicated to those ignited minds who always challenges their limits."

CERTIFICATE

This is to certify that the work contained in the project entitled **"An Application of Fourier Analysis to understand Musical Sounds and Principle of Additive Synthesis"** submitted by Jaminur Ali Akhtar (Regd. No: 18034408) for the award of the degree of Bachelor of Science, Under Gauhati University carried out by him under my direct supervision and guidance.

I considered that the project has reached the standards and fulfilling the Requirements of the rules and regulation relating to the nature of the degree.

Date:

Place:

(Dr. Dhiraj Kumar Das)

HOD of Mathematics

Jawaharlal Nehru College, Boko

Declaration

I hereby declare that the featuring of this project is a result of my own work under the guidance of Associate Professor **Dr. Dhiraj Kumar Das** of Department of Mathematics, Jawaharlal Nehru College Boko. I'd like to declare that no part of this project has been previously submitted to this college to the best of my knowledge.

Place: Boko

Date: - 23/08/2021

Jaminur Ali Akhtar

(Signature of the student)

Acknowledgement

I do hereby thank my mentor **Dr. Dhiraj Kumar Das** for offering assistance on my project. I also take this opportunity to thank all the teachers of Department of mathematics of Jawaharlal Nehru College, Boko for being always available.

I would also like to thank and express my deep appreciation to my well wishers for their personal support and encouragement in doing this project.

Jaminur Ali Akhtar
23/08/2021

ABSTRACT: Music though it seems a field of art, however the sounds that makes up a music carries relationship that are purely mathematical in nature. From sounds of guitar to sounds of electronic keyboard the frequencies carry a harmonic relationship and the sound wave of musical sound are periodic functions which can be expressed by a Fourier Series. In this project we will focus on musical sounds, learn about harmonics, Fourier Series, see the harmonic content of a square wave and using Additive Synthesis we will create a Saw and Square wave which are widely used in music production.

KEYWORDS: Frequency, harmonic frequency, periodic functions, Fourier Series, Additive Synthesis.

CONTENTS

INTRODUCTION: The subject of music and mathematics though distant however carries a relationship that goes hand in hand. In the course of history various personalities have successfully discovered some of the many relationships that defines what type of sound makes up a music. For example, In 600 BC pythagoreans were able to discover about harmonic frequency and their work was completed by Galileo Galilei and Mariene Mersenne by discovering the 'laws of vibrating strings' in 1636.

In 1807 Jean-Baptise Joseph Fourier introduced a series to solve the heat equation. This is today called a Fourier Series. It is defined as a periodic function of harmonically related sinusoids. Today the development from the idea of fourier series has changed the world and brought the domain of sound design and music production from analog to digital world. Knowledge from the fourier series has made possible to design sounds in computer for music production purposes.

First of all let us understand the basic concepts related to sounds in the next chapter 1.

Chapter 1: Basics of Sounds

1.1 Sound: Sound is a pressure wave which is created by a vibrating object. These vibrations set particles in the surrounding medium in vibrational motion, thus transporting energy through the medium.

1.2 Nature of Sound: Since the particles are moving in parallel direction to the wave movement, the sound wave is reffered to as a longitudinal wave.
The result of longitudinal wave is the creation of compression and rarefractions within the air.

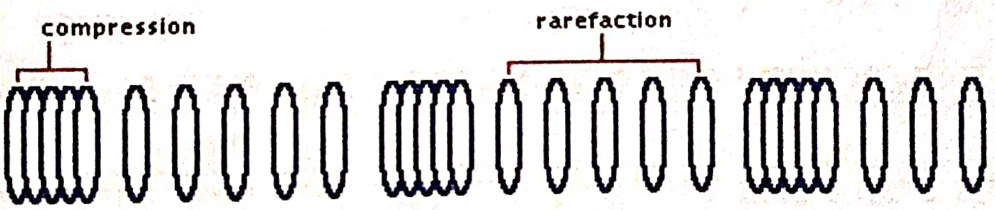


Figure 1: Longitudinal Wave

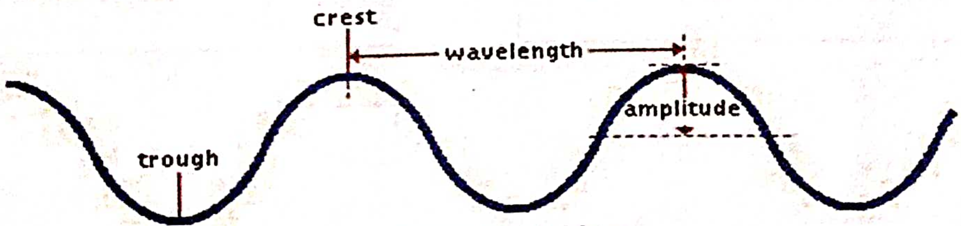
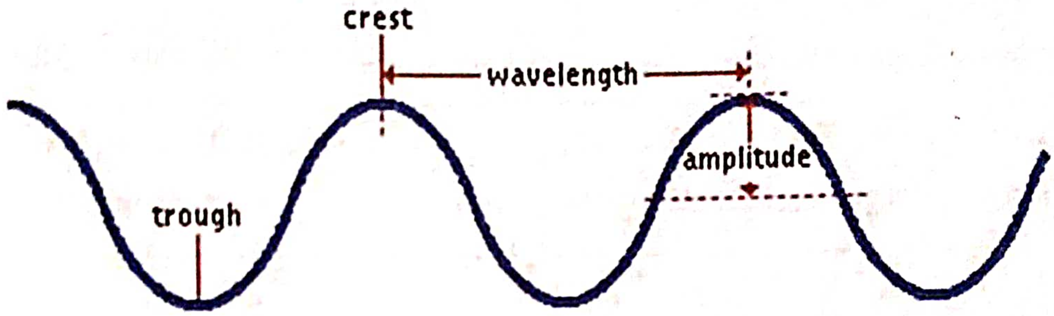


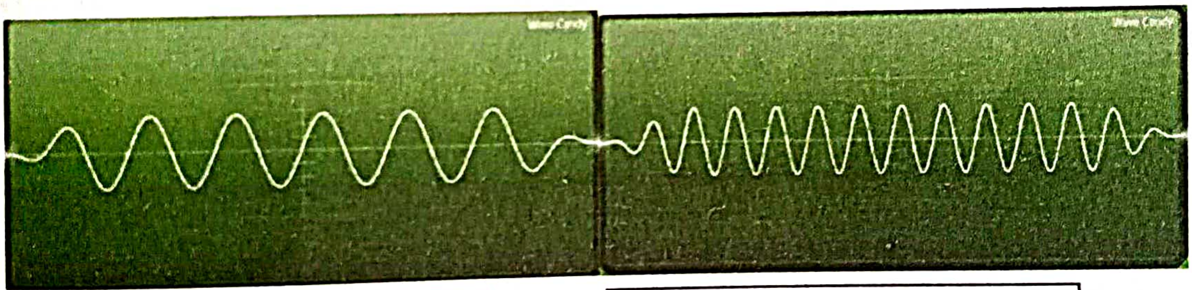
Figure 2: Transverse Wave

1.3 Properties of Sound Wave:

Sound wave can be described by 5 characteristics

1) Wavelength:- The distance between two successive crest or trough is called a wavelength. It is the minimum distance in which a sound wave repeats itself.

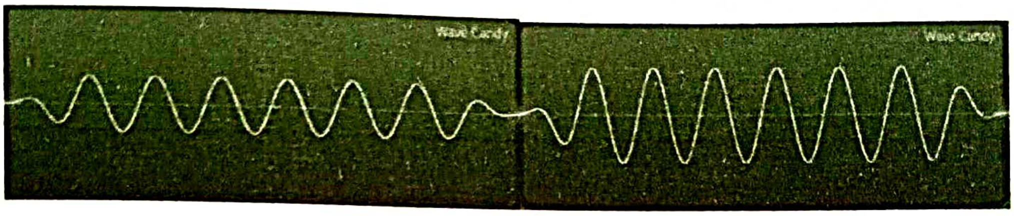




A4=440Hz. High Wavelength.

A5=880Hz. Low Wavelength.

2) Amplitude :- Amplitude is the size of the vibration. It is the displacement of the wave from the mean position. This determines how loud the sound is.

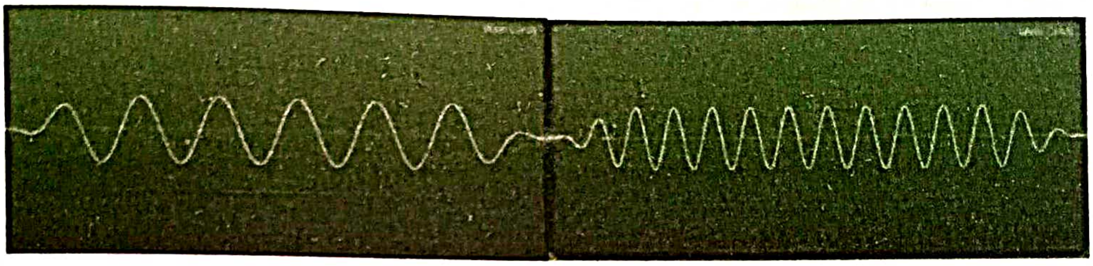


A4=440Hz. Low amplitude	A4=440Hz. High amplitude

Amplitude is important when balancing and controlling the loudness of sounds, such as with the volume control of a speaker or mixer. It is also the origin of the word amplifier, a device which increases the amplitude of a waveform.

3) Time Period :- The time taken to complete one-vibration is known as time period.

4) Frequency or Pitch :- The number of complete waves or cycles produced in one second is called frequency of the wave.



A4=440Hz. Low frequency

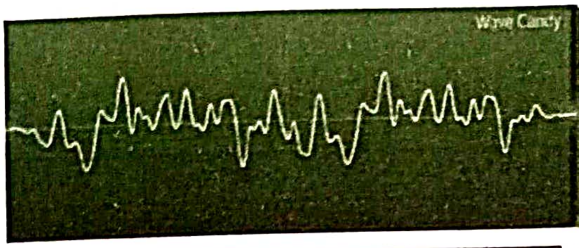
A5=880Hz. High frequency

Pitch is how high or low scale the sound is when we hear it. So A4 is low pitch when compared to A5 which is comparatively high when we hear it.

5) Velocity or Speed :- The distance travelled by a sound wave in 1 second is called velocity or speed of the sound wave.

Chapter 2: Musical Sounds

2.1 Musical Sound:- The difference between a non-musical sound and a musical sound is that musical sounds have periodic vibration. Due to which we perceive these sounds as Pitch.



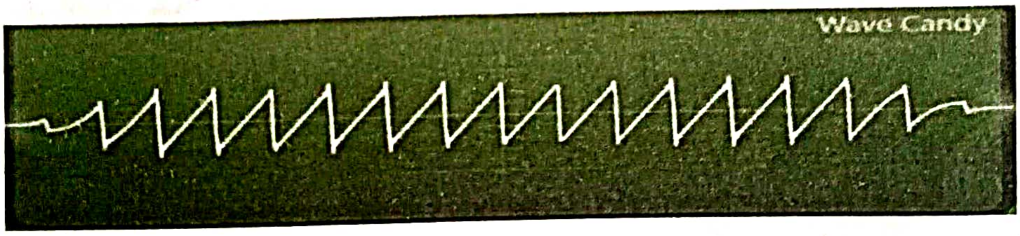
A4 note on an electric guitar. (Notice the waveform is repeating)

Noise wave.

Examples of Sound Wave used in music production

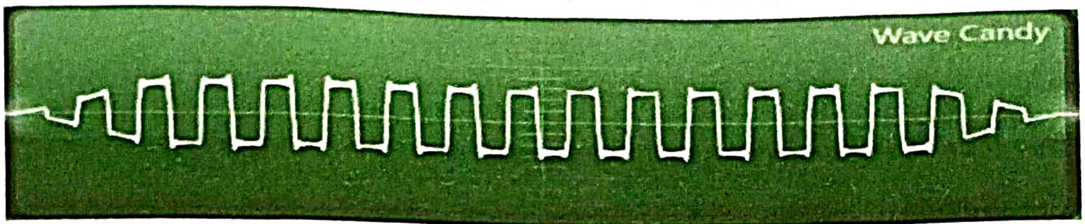
1) Saw wave of amplitude 'a' and base frequency 'f'.

$$x(t) = a\left(\frac{1}{2} - \frac{1}{\pi}\sum_{1}^{\infty}(-1)^{k}\frac{\sin\left(\frac{2\pi kft}{1}\right)}{k}\right)$$



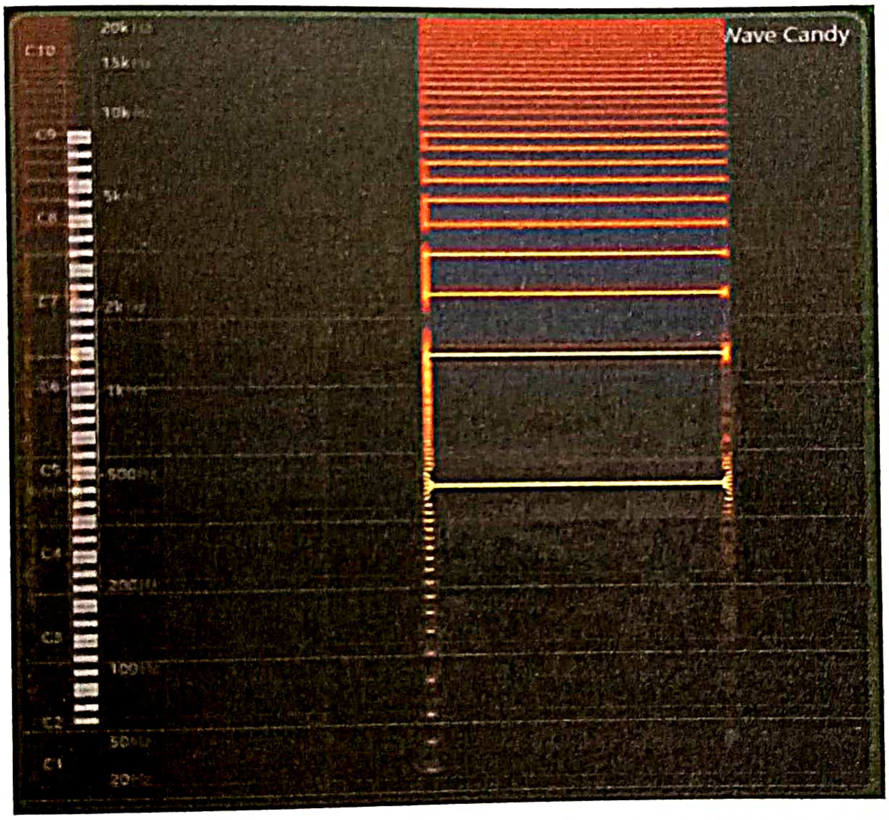
2) Square wave of amplitude '1' and frequency 'f'.

$$x(t) = \frac{4}{\pi}\sum_{1}^{\infty}\frac{\sin(2\pi(2k-1)ft)}{2k-1}$$



On the other hand the sound of aeroplane, utensils, vehicle due to their non-periodicity seems noise to human ear.

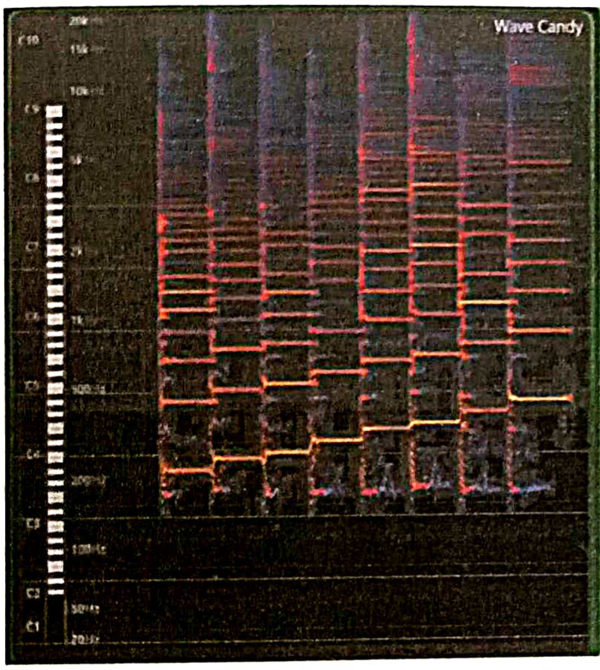
2.2 Understanding the Spectrogram:

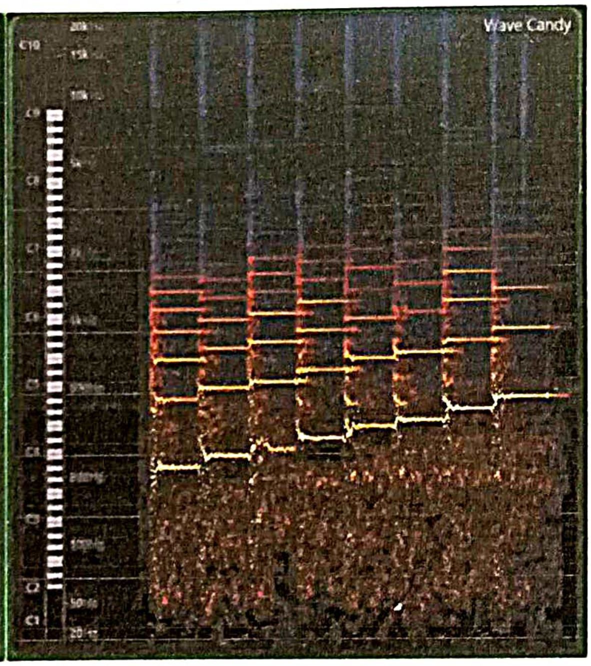


Displays a moving trace of audio signal. Colour hue represents intensity, vertical position represent frequency (20 Hz to 20kHz, bottom to top is represented) and horizontal position represent time.

2.3 Harmonics and Overtones:

In our daily life we have heard in various programs that along with human voice different musical instruments some in high pitch some in low pitch are played together. They sound distinct yet are complety in harmony.





Spectrum of playing A minor scale on guitar

Spectrum of playing A minor Scale on Piano.

If we analyse the spectrum carefully, we see that they are identical. The base frequency (lowest line in the spectrum) is same for both the frequency of the instruments, followed by next upper frequency, hence so on. So they sound good together or are in harmony.

The reason they sound different is because of guitar co-ntaining more high frequencies (visible in the spectrum). These higher frequencies that gives an instrument its characteristic sound are called harmonics or overtones.

These frequencies carries a relationship which is

$$f_n = n f_0 , \; n \in \mathbb{N}$$

where

f_0 = base frequency

f_n = n^{th} harmonic frequency.

Chapter 3: Mathematics of Musical Sounds

3.1 Fourier Series and Musical Sound:

Musical sounds are periodic in nature. A signal which is repetitive is a periodic function of time. Any periodic function of time $f(x)$ can be expressed by an infinite series called Fourier Series.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \rightarrow (1)$$

where,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$f(x)$ satisfies the following conditions

(i) f is well defined and single valued except possibly at finite number of points.

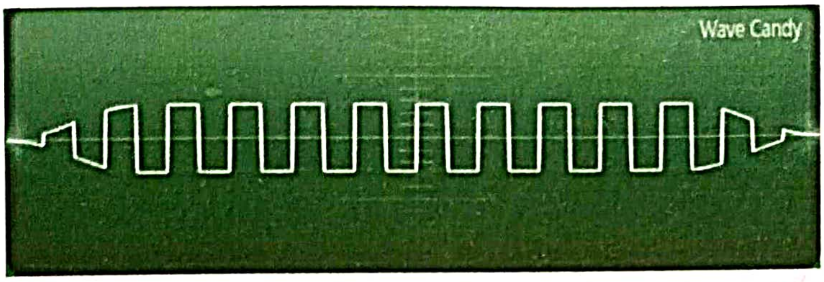
(ii) f must possess only a finite number of discontinuities in period $[-\pi, \pi]$.

(iii) f must have a finite number of positive maxima and negative maxima in period $[-\pi, \pi]$.

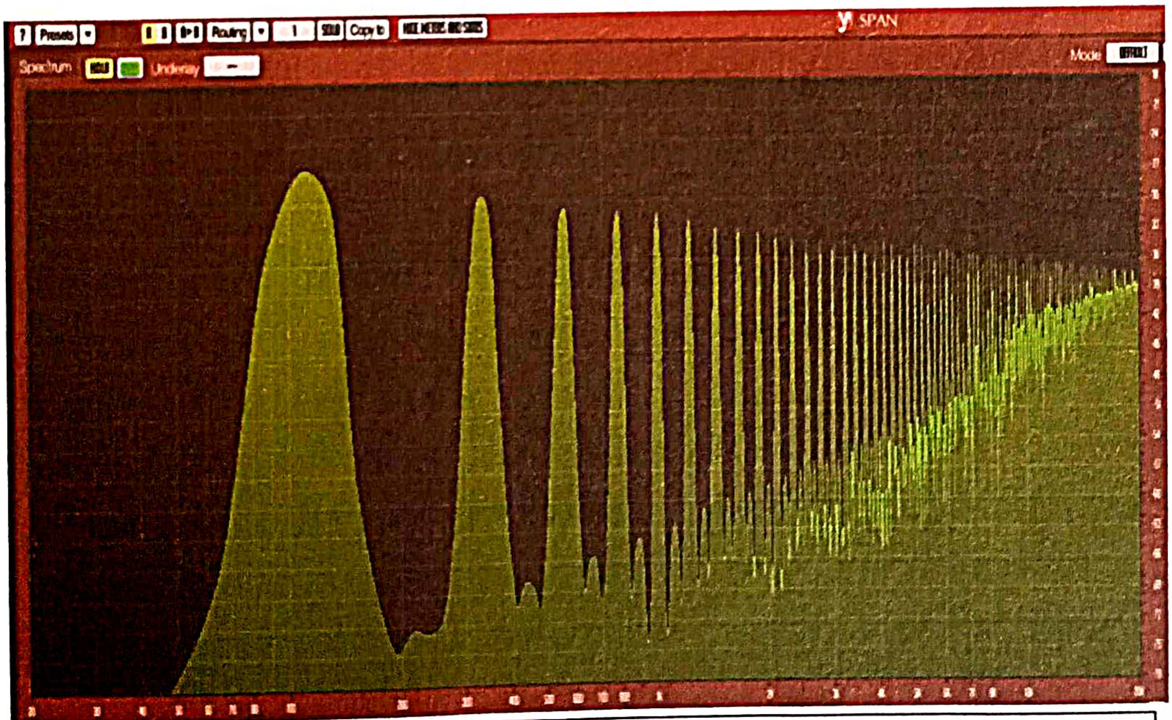
3.2 Fourier Analysis of a Square Wave :

Square wave is one of the most widely used wave to create musical sounds. It is used widely in making lead sound to sounds of chord and various other sounds. Square wave is behind the sound of a fuzz distortion guitar.

For this purpose, we record a square wave at 110 Hz

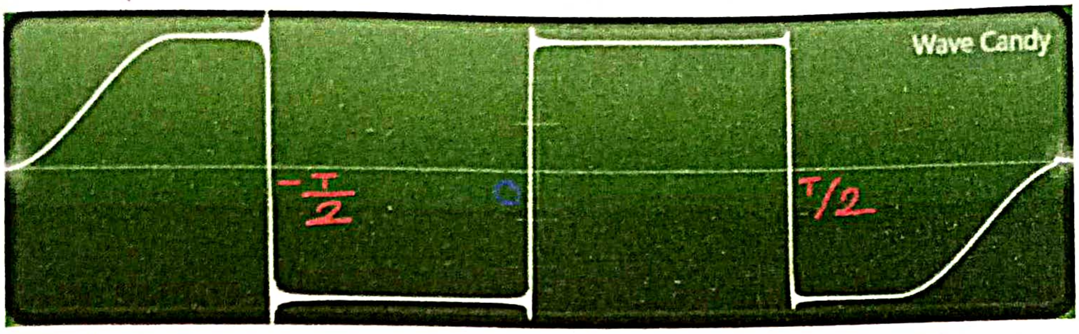


The waveform is repeating. However we see from the frequency-amplitude graph, the signal is much more complex.



Each peak in the graph represent a harmonic.

Let us break the Square Wave into its Component harmonics.



The function of the wave for one period can be written as

$$f(t) = \begin{cases} -A, & -\frac{T}{2} < t < 0 \\ A, & 0 < t < \frac{T}{2} \end{cases}$$

As the waveform is symmetric about the origin, the function is odd and hence $a_0 = a_n = 0$, and

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{T}\left[\int_{-\frac{T}{2}}^{0} f(t)\sin(n\omega_0 t)\,dt + \int_{0}^{\frac{T}{2}} f(t)\sin(n\omega_0 t)\,dt\right]$$

$$= \frac{2A}{T}\left\{\left[\frac{\cos n\omega_0 t}{n\omega_0}\right]_{-T/2}^{0} + \left[-\frac{\cos n\omega_0 t}{n\omega_0}\right]_{0}^{T/2}\right\}$$

$$= \frac{2A}{T}\cdot\frac{1}{n\omega_0}\left\{\left[1-\cos\left(n\omega_0\frac{T}{2}\right)\right] + \left[1-\cos\left(n\omega_0\frac{T}{2}\right)\right]\right\}$$

$$= \frac{2A}{n\omega_0 T}\cdot 2\left\{1-\cos\left(n\frac{\omega_0 T}{2}\right)\right\}$$

$$= \frac{4A}{n\omega_0 T}\left\{1-\cos n\pi\right\} \quad \left[\because \omega_0 = \frac{2\pi}{T}\right]$$

$$\therefore\ b_n = \begin{cases} 0, & n \text{ is even} \\ \frac{8A}{n\omega_0 T}, & n \text{ is odd} = \frac{8A}{n\cdot 2\pi} = \frac{4A}{n\pi} \end{cases}$$

Substituting the value of b_n in (1), we get

$$f(t) = \frac{4A}{\pi}\left[\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \dots\right]$$

where $\omega_0 = \frac{2\pi}{T} = 2\pi f$

∴ We can see that a simple square wave is the sum of odd harmonics of sine.

Similarly every sound in nature is made of sine wave. It is the type of relation that determines the sound will be musical or note.

So, if we think the other way then is it possible that we can add sine waves in varying numbers and create countably infinite sounds?

The question turned out to be yes, and this gives rise to Additive Synthesis.

Chapter 4 : Additive Synthesis

4.1 Introduction: Additive Synthesis is a Sound generation technique by adding Sine Waves of different amplitude and frequencies. Theoritically every Sound in the nature can be created by additive Synthesis. With the development of digital electronics and powerful computers Additive Synthesis found its place in the music Industry. Using Software such as Imag-Line Harmour, Image-Line Morphine one can add upto 128 Sine Waves of varying frequency and amplitude to Create Sounds of very simple to Complex Waveform.

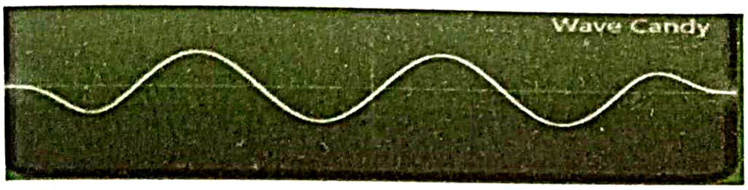
So, let us Create two most widely used Sound Wave in electronic music using the Additive Synthesis.

4.2 Making a Square Wave:

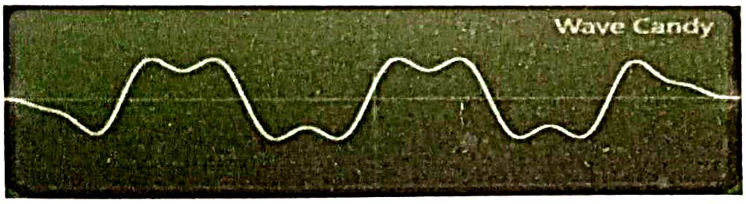
First we start with a Sine wave and then we will add the odd harmonics.

Let N be the number of Sine waves.

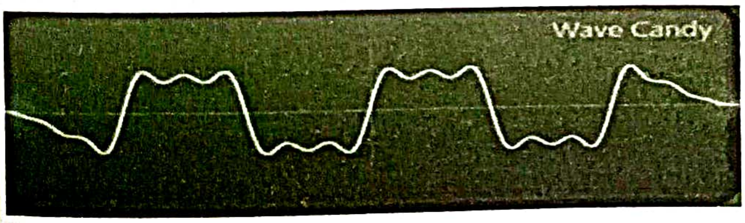
$N=1: f(t) = \frac{4}{\pi} \sin(2\pi f t)$



$N=2: f(t) = \frac{4}{\pi}\left[\sin(2\pi f t) + \frac{1}{3}\sin(3 \cdot 2\pi f t)\right]$



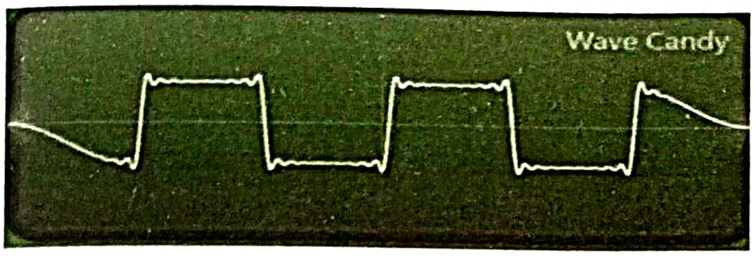
$N=3: f(t) = \frac{4}{\pi}\left[\sin(2\pi f t) + \frac{1}{3}\sin(3 \cdot 2\pi f t) + \frac{1}{5}\sin(5 \cdot 2\pi f t)\right]$



Similarly we add odd harmonics 11 times

N= 12:

$$f(t) = \frac{4}{\pi}\left[\sin(2\pi f t) + \frac{1}{3}\sin(3\cdot 2\pi f t) + \frac{1}{5}\sin(5\cdot 2\pi f t) + \cdots \right.$$
$$\left. \cdots + \frac{1}{25}\sin(25\cdot 2\pi f t)\right]$$



Thus we see that as N Increases the wave is becoming more and more Square. With N> 100 the wave becomes identically Square.

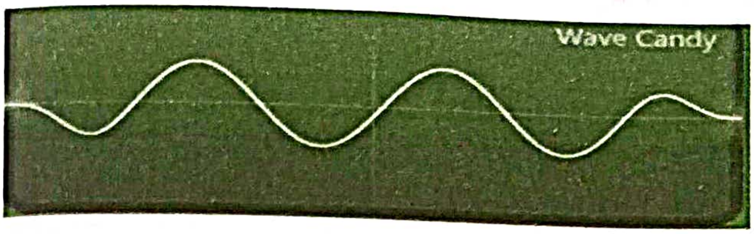
With the Square Wave ready we just need a ADSR envelope and effects to start shaping the sound and making our favourite music.

4.3 Making a Saw Wave:

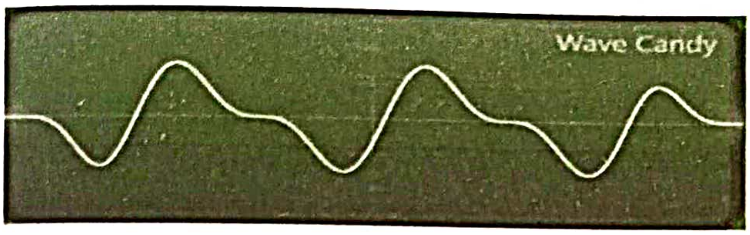
Saw Wave is characterized by a buzzy sound and a rich source of harmonics for various sound design purposes. Unlike square wave, triangle wave, pulse wave, saw wave contains all the harmonics of its fundamental frequency. It is the prime sound behind Electronic Dance Music (EDM) such as trance, progressive house, etc..

Let N be the number of Sine waves.

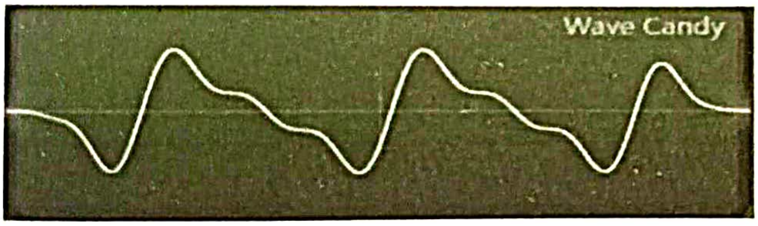
N=1: $f(t) = \frac{1}{\pi} \sin(2\pi f t)$



N=2: $f(t) = \frac{1}{\pi}\left[\sin(2\pi f t) + \frac{1}{2}\sin(2\times 2\pi f t)\right]$



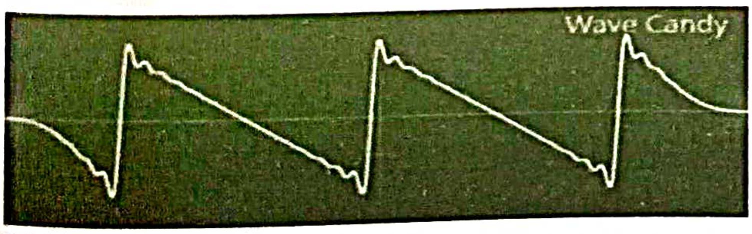
N=3: $f(t) = \frac{1}{\pi}\left[\sin(2\pi f t) + \frac{1}{2}\sin(2\times 2\pi f t) + \frac{1}{3}\sin(3\times 2\pi f t)\right]$



Adding to 16th harmonics

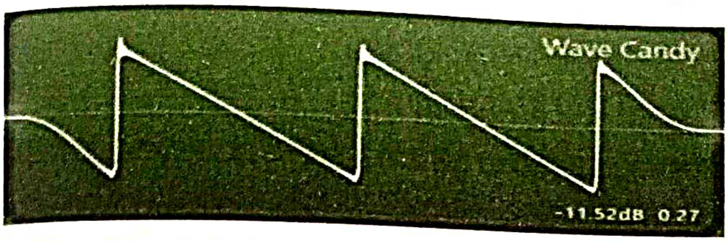
N=16: $f(t) = \frac{1}{\pi}\left[\sin(2\pi f t) + \frac{1}{2}\sin(2\times 2\pi f t) + \frac{1}{3}\sin(3\times 2\pi f t) + \dots \right.$

$\left. \dots + \frac{1}{16}\sin(16\times 2\pi f t)\right]$



Thus, we see that as N increases the wave turns to saw.

For N=100+ the wave turns identically saw.



Thus, with our saw wave ready we just need a ADSR envelope and effects to design the sound to our purpose and make hit music.

Chapter 5 : Conclusion

It is quite fascinating how the knowledge of harmonics and the Fourier Series initially developed to solve the heat equation has changed how we understand musical sounds and inspired Mathematicians and engineers to build upon the idea and changed how we create sound today. With the development of Digital Signal Processing and availability of powerful computers, we are able to put into effect various sound designing technique such as FM Synthesis, Subtractive Synthesis, Wavetable Synthesis, which has brough a revolution in the music industry since the beginning of 21st Century. Just listening to music gives us immense pleasure and when we as a student of mathematics finds out the mathematics behind it, motivates us to learn more about music and do further research. Due to lack of time it was not possible to show many things in this project, however in the future I would like to gain further knowledge in this field.

Chapter 6: Bibliography

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(iii) Voxengo SPAN.
(iv) 3xOSC by Image-Line
(v) DSK Dynamic guitar.
(vi) LABS
(vii) Harmour by Image-Line.

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(ii) https://www.howmusicworks.org/100/Sound-and-music.
(iii) https://en.wikipedia.org/wiki/Additive_Synthesis

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