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**3 (Sem-3 /CBCS) MAT HC 1**

**2021**

**(Held in 2022)**

# MATHEMATICS

(Honours)

Paper : MAT-HC-3016

***( Theory of Real Functions )***

*Full Marks : 80*

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

1. Answer the following as directed : $1\times10=10$

(a) Find $\lim\limits_{x\to2}\dfrac{x^3-4}{x^2+1}$

(b) Is the function $f(x)=x\sin\left(\dfrac{1}{x}\right)$ continuous at $x=0$ ?

(c) Write the cluster points of $A=(0,1)$.

Contd.

(d) If a function $f:(a,\infty)\to\mathbb{R}$ is such that $\lim_{x\to\infty} xf(x)=L$, where $L\in\mathbb{R}$, then $\lim_{x\to\infty} f(x)=?$

(e) Write the points of continuity of the function $f(x)=\cos\sqrt{1+x^2}$, $x\in\mathbb{R}$.

(f) "Every polynomial of odd degree with real coefficients has at least one real roof." Is this statement true ***or*** false ?

(g) The derivative of an even function is ________ function. *(Fill in the blank)*

(h) Between *any two* roots of the function $f(x)=\sin x$, there is at least ________ root of the function $f(x)=\cos x$.

*(Fill in the blank )*

(i) If $f(x)=\left|x^3\right|$ for $x\in\mathbb{R}$, then find $f'(x)$ for $x\in\mathbb{R}$.

(j) Write the number of solutions of the equation $ln(x)=x-2$.



2. Answer the following questions : 2×5=10

(a) Show that $\lim_{x\to0}(x+sgn(x))$ does not exist.

(b) Let $f$ be defined for all $x\in\mathbb{R}$, $x\neq3$ by $f(x)=\dfrac{x^2+x-12}{x-3}$. Can $f$ be defined at $x=3$ in such a way that $f$ is continuous at this point ?

(c) Show that $f(x)=x^2$ is uniformly continuous on $[0,a]$, where $a>0$.

(d) Give an example with justification that a function is 'continuous at every point but whose derivative does not exist everywhere'.

(e) Suppose $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2\sin\frac{1}{x^2}$, for $x\neq0$ and $f(0)=0$. Is $f'$ bounded on $[-1,1]$ ?

3. Answer **any four** parts : 5×4=20

(a) If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then prove that $f$ is bounded on some neighbourhood of $c$.

(b) Let $f(x) = |2x|^{-\frac{1}{2}}$ for $x \neq 0$. Show that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = +\infty.$$

(c) Show that the function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$.

(d) Give an example to show that the product of two uniformly continuous function is not uniformly continuous on $\mathbb{R}$.

(e) Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. If $f'$ is positive on $[a, b]$, then prove that $f$ is strictly increasing on $[a, b]$.

(f) Evaluate —

$$\lim_{x\to 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right)$$



4. Answer **any four** parts : 10×4=40

(a) Let $f : A \to \mathbb{R}$ and let $c$ be a cluster point of $A$. Prove that the following are equivalent.

(i) $\lim_{x\to c} f(x) = l$

(ii) For every sequence $(x_n)$ in $A$ that converges to $c$ such that $x_n \neq c$ for all $x \in \mathbb{N}$, the sequence $(f(x_n))$ converges to $l$. 10

(b) (i) Give examples of functions $f$ and $g$ such that $f$ and $g$ do not have limits at a point $c$ but such that both $f+g$ and $fg$ have limits at $c$. 6

(ii) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c$ be a cluster point of $A$. If $\lim_{x\to C} f(x)$ exists and if $|f|$ denotes the function defined for $x \in A$ by $|f|(x) = |fx|$, Proof that

$$\lim_{x\to c} |f|(x) = \left| \lim_{x\to c} f(x) \right|$$ 4

(c) Prove that the rational functions and the sine functions are continuous on $\mathbb{R}$. 10

(d) (i) Let $I$ be an interval and let $f: I \to \mathbb{R}$ be continuous on $I$. Prove that the set $f(I)$ is an interval. 5

(ii) Show that the function $f(x) = \dfrac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. 5

(e) State and prove maximum-minimum theorem. 2+8=10

(f) (i) If $f: I \to \mathbb{R}$ has derivative at $c \in I$, then prove that $f$ is continuous at $c$. Is the converse true ? Justify. 6



(ii) If $r$ is a rational number, let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Determine those values of $r$ for which $f'(0)$ exists. 4

(g) State and prove Mean value theorem. Give the geometrical interpretation of the theorem. (2+5)+3=10

(h) State and prove Taylor's theorem.

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3 (Sem–3/CBCS) MAT HC 3

**2021**

**(Held in 2022)**

# MATHEMATICS

(Honours)

Paper : MAT–HC–3036

***(Analytical Geometry)***

*Full Marks : 80*

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

1. Answer the following questions : 1×10=10

   *(i)* What is the nature of the conic represented by

   $$4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0 \ ?$$

   *(ii)* Define skew lines.

(iii) Under what condition

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

may represents a pair of parallel straight lines ?

(iv) If the axes are rectangular, find the direction cosines of the normal to the plane $x + 2y - 2z = 9$.

(v) Write down the conditions under which the general equation of second degree

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$

represents a sphere.

(vi) If $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is a generator of the cone represented by the homogeneous equation $f(x, y, z)$, then what is the value of $f(l, m, n)$ ?

(vii) What is meant by diametral plane of a conicoid ?



(viii) Find the equation of the line $\frac{x}{a} + \frac{y}{b} = 2$, when the origin is transferred to the point $(a, b)$.

(ix) Find the point on the conic $\frac{8}{r} = 3 - \sqrt{2}\cos\theta$ whose radius vector is 4.

(x) What is the polar equation of a circle when the pole is at the centre ?

2. Answer the following questions : 2×5=10

(a) Write down the equation to the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + cz^2 = 1$.

(b) Transform the equation $x^2 - y^2 = a^2$ by taking the perpendicular lines $y - x = 0$ and $y + x = 0$ as coordinate axes.

(c) If $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are the extremities of any focal chord of the parabola $y^2 = 4ax$, then prove that $t_1 t_2 = -1$.

(d) Find the centre and foci of the hyperbola $x^2 - y^2 = a^2$.

(e) Find where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ meets the plane $x+y+z=3$.

3. Answer ***any four***: 5×4=20

(a) If by transformation from one set of rectangular axes to another with the same origin the expression $ax+by$ changes to $a'x'+b'y'$, prove that $a^2+b^2 = a'^2+b'^2$.



(b) Prove that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a pair of parallel straight lines, if $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$.

(c) Find the condition that line

$$\frac{l}{r} = A\cos\theta + B\sin\theta$$

may touch the conic $\frac{l}{r} = 1 - e\cos\theta$.

(d) Find the equation to the plane which cuts $x^2+4y^2-5z^2=1$ in a conic whose centre is the point (2,3,4).

(e) Show that the equation to the cone whose vertex is origin and base is $z=k,\ f(x,y)=0$ is $f\left(\frac{kx}{z}, \frac{ky}{z}\right)=0$.

(f) A variable plane is at a constant distance $p$ from the origin and meets the axes, which are rectangular in $A$, $B$, $C$. Through $A$, $B$, $C$ planes are drawn parallel to the coordinate planes, show that locus of their point of intersection is given by $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

4. Answer the following questions : 10×4=40

(a) Find the point of intersection of the lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$.

(b) Show that the equation

$$9x^2-24xy+16y^2-18x-101y+19=0$$

represents a parabola and it can be reduced to the standard form $Y^2=3X$. Find the coordinates of the vertex and the focus.



(c) Prove that the sum of the reciprocals of two perpendicular focal chords of a conic is constant.

(d) Show that the ortho-centre of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ is given by $\frac{x}{l}=\frac{y}{m}=\frac{a+b}{am^2-2hlm+bl^2}$

(e) Find the condition that the plane $lx+my+nz=p$ may touch the conicoid $ax^2+by^2+cz^2=1$. Verify that the plane $2x-2y+8z=9$ touches the ellipsoid $x^2+2y^2+3z^2=9$.

(f) Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $y+z=0$, $z+x=0$, $x+y=0$, $x+y+z=a$ is $\frac{2a}{\sqrt{6}}$ and that the three lines of shortest distance intersect at the point $x=y=z=-a$.

(g) Find the equation to the cylinder generated by the lines drawn through the points of the circle

$x+y+z=1, x^2+y^2+z^2=4$ which are

parallel to the line $\dfrac{x}{2}=\dfrac{y}{-1}=\dfrac{z}{2}$.

(h) A variable plane is parallel to the given plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

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