Total number of printed pages–7

3 (Sem–6/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–6026

(Partial Differential Equations)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** : 1×7=7

   (i) The equation of the form

   $P_p + Q_q = \mathbb{R}$ is known as

   (a) Charpit's equation

   (b) Lagrange's equation

   (c) Bernoulli's equation

   (d) Clairaut's equation

   (Choose the correct answer)

Contd.

(ii) How many minimum no. of independent variables does a partial differential equation require?

(iii) Find the degree and order of the equation

$$\frac{\partial^3 z}{\partial x^3}+\left(\frac{\partial^3 z}{\partial x \partial y^2}\right)^2+\frac{\partial z}{\partial y}=\sin(x+2y)$$

(iv) Which method can be used for finding the complete solution of a non-linear partial differential equation of first order

(a) Jacobi method

(b) Charpit's method

(c) Both (a) and (b)

(d) None of the above

(Choose the correct answer)

(v) State *True* **Or** *False* :

The equation

$$u_{xx}+u_{yy}+u_{zz}=0$$

is an Hyperbolic equation.

(vi) Fill in the blanks :

$$\left(\frac{\partial z}{\partial x}\right)^2+2\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}+z=0$$

is a ________ ________ order partial differential equation.

(vii) The characteristic equation of $yu_x + xu_y = u$ is

(a) $\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$

(b) $\frac{dx}{y} = \frac{dy}{x} = \frac{du}{u}$

(c) $\frac{dx}{u} = \frac{dy}{x} = \frac{du}{y}$

(d) None of the above

(Choose the correct answer)

(viii) State *True* **Or** *False*

$xu_x + yu_y = u^2 + x^2$ is a semi-linear partial differential equation.

(ix) Fill in the blanks :

A solution $z = z(x, y)$ when interpreted as a surface in 3-dimensional space is called ________ ________ .

(x) The partial differential equation is elliptical if

(a) $B^2 - 4AC > 0$

(b) $B^2 - 4AC \geq 0$

(c) $B^2 - 4AC \leq 0$

(d) $B^2 - 4AC < 0$.

(Choose the correct answer)

2. Answer ***any four*** : $2 \times 4 = 8$

(i) Define quasi-linear partial differential equation and give *one* example.

(ii) Show that a family of spheres $(x-a)^2 + (y-b)^2 = r^2$ satisfies the partial differential equation $z^2(p^2 + q^2 + 1) = r^2$

(iii) Eliminate the constants a and b from $z = (x+a)(y+b)$.

(iv) Determine whether the given equation is hyperbolic, parabolic or elliptic

$$u_{xx} - 2u_{yy} = 0.$$

(v) Solve the differential equation $p + q = 1$.

(vi) Explain the essential features of the "Method of separation of variables".

(vii) Mention when Charpit's method is used. Name a disadvantage of Charpit's method.

(viii) What is the classification of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = e^y$$

3. Solve *any three* : 5×3=15

(i) Form a partial differential equation by eliminating arbitrary functions f and F from $y = f(x-at) + F(x+at)$.

(ii) Solve

$$y^2 p - xyq = x(z-2y)$$

(iii) Find the integral surface of the linear partial differential equation

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains the straight line $x+y=0,\ z=1$.

(iv) Find the solution of the equation $z = pq$ which passes through the parabola $x=0,\ y^2 = z$.

(v) Find a complete integral of the equation $x^2p^2 + y^2q^2 = 1$.

(vi) Reduce the equation $yu_x + u_y = x$ to canonical form and obtain the general solution.

(vii) Apply the method of separation of variables $u(x, y) = f(x)g(y)$ to solve the equation $u_x + u = u_y$,

$u(x, 0) = 4e^{-3x}$.

(viii) Determine the general solution of

$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$.

4. Answer ***any three*** : 10×3=30

(i) Solve $(p^2 + q^2)y - qz = 0$ by Jacobi method.

(ii) Solve $z^2 = pqxy$ by Charpit's method.

(iii) Find the general solution of the differential equation

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x + y)z$$

(iv) Solve

$$(mz - ny)p + (nx - lz)q = ly - mx$$

(v) Use $v = \ln u$ and $v = f(x) + g(y)$ to solve the equation

$$x^2u_x^2 + y^2u_y^2 = u^2.$$

(vi) Find the solution of the equation

$$z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$$

which passes through the x axis.

(vii) Find the canonical form of the equation

$$y^2 u_{xx} - x^2 u_{yy} = 0.$$

(viii) Classify the second order linear partial differential equation with example.

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Total number of printed pages–8

3 (Sem–6/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–6016

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** questions from the following : $1 \times 7 = 7$

(a) If c is any nth root of unity other than unity itself, then value of $1 + c + c^2 + \cdots + c^{n-1}$ is

(i) $2n\pi$

(ii) 0

(iii) -1

(iv) None of the above

(Choose the correct answer)

Contd.

(b) The square roots of $2i$ is

(i) $\pm(1+i)$

(ii) $\pm(1-i)$

(iii) $\pm\frac{1}{\sqrt{2}}\left(1-i\sqrt{2}\right)$

(iv) None of the above

(Choose the correct answer)

(c) A composition of continuous function is

(i) discontinuous

(ii) itself continuous

(iii) pointwise continuous

(iv) None of the above

(Choose the correct answer)

(d) The value of $Log(-ei)$ is

(i) $\frac{\pi}{2}-i$

(ii) i

(iii) $1-\frac{\pi}{2}i$

(iv) None of the above

(Choose the correct answer)

(e) The power expression of $\cos z$ is

(i) $\frac{e^{z}+e^{-z}}{2}$

(ii) $\frac{e^{iz}+e^{-iz}}{2}$

(iii) $\frac{e^{iz}+e^{-iz}}{2i}$

(iv) None of the above

(Choose the correct answer)

(f) The Cauchy-Riemann equation for analytic function $f(z)=u+iv$ is

(i) $u_x = v_y$, $u_y = -v_x$

(ii) $u_x = -v_y$, $u_y = v_x$

(iii) $u_{xx} + v_{yy} = 0$

(iv) None of the above

(Choose the correct answer)

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(g) If $w(t)=u(t)+iv(t)$, then $\frac{d}{dt}[w(t)]^2$ is equal to

(i) $2[u(t)+iv(t)]$

(ii) $2w'(t)$

(iii) $2w(t)w'(t)$

(iv) None of the above

(Choose the correct answer)

(h) What is Laplace's equation ?

(i) What is extended complex plane ?

(j) What is Jordan arc ?

2. Answer ***any four*** questions from the following : 2×4=8

(a) Write principal value of $arg\left(\frac{i}{-1-i}\right)$.

(b) If $f(z) = x^2 + y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$, then write $f(z)$ in terms of z.

(c) Use definition to show that
$$\lim_{z \to z_0} \bar{z} = \bar{z}_0.$$

(d) Find the singular point of
$$f(z) = \frac{z^2 + 3}{(z+1)(z^2+5)}.$$

(e) If $f'(z) = 0$ everywhere in a domain D, then prove that $f(z)$ must be constant throughout D.

(f) Evaluate $f'(z)$ from definition, where $f(z)=\frac{1}{z}$.

(g) If $f(z)=\frac{z}{\bar{z}}$, find $\lim_{z\to 0} f(z)$, if it exists.

(h) Write the function $f(z)=z+\frac{1}{z}\ (z\neq 0)$ in the form $f(z)=u(r,\theta)+iv(r,\theta)$.

3. Answer ***any three*** questions from the following : 5×3=15

(a) If z_1 and z_2 are complex numbers, then show that
$\sin(z_1+z_2)=\sin z_1 \cos z_2+\cos z_1 \sin z_2$.

(b) Show that exp. $(2\pm 3\pi i)=-e^2$.

(c) Sketch the set $|z-2+i|\leq 1$ and determine its domain.

(d) Let C be the arc of the circle $|z|=2$ from $z=2$ to $z=2i$, that lies in the 1st quadrant, then show that

$$\left|\int_C \frac{z-2}{z^2+1}\,dz\right|\leq \frac{4\pi}{15}$$

(e) Evaluate $\int_C \frac{dz}{z}$, where C is the top half of the circle $|z|=1$ from $z=1$ to $z=-1$.

(f) If $f(z)=e^z$, then show that it is an analytic function.

(g) If $f(z)=\frac{z+2}{z}$ and C is the semi circle $z=2e^{i\theta}$, $(0 \le \theta \le \pi)$, then evaluate $\int_C f(z)\, dz$.

(h) Find all values of z such that $e^z = -2$.

4. Answer **any three** questions from the following : 10×3=30

(a) State and prove Cauchy-Riemann equations of an analytic function in polar form.

(b) Suppose that $f(z)=u(x, y)+iv(x, y)$, $(z=x+iy)$ and $z_0=x_0+iy_0$, $w_0=u_0+iv_0$, then prove that if $\lim\limits_{(x, y)\to(x_0, y_0)} u(x, y)=u_0$ and $\lim\limits_{(x, y)\to(x_0, y_0)} v(x, y)=v_0$ then $\lim\limits_{z\to z_0} f(z)=w_0$ and conversely.

(c) If the function $f(z) = u(x, y) + iv(x, y)$ is defined by means of the equation

$$f(z) = \begin{cases} \dfrac{\bar{z}^e}{z}, & \text{when } z \neq 0 \\ 0, & \text{when } z = 0, \end{cases}$$

then prove that its real and imaginary parts satisfies Cauchy-Riemann equations at $z = 0$. Also show that $f'(0)$ fails to exist.

(d) If the function

$f(z) = u(x, y) + iv(x, y)$ and its conjugate $\bar{f}(z) = u(x, y) - iv(x, y)$ are both analytic in a domain *D*, then show that $f(z)$ must be constant throughout *D*.

(e) If f be analytic everywhere inside and on a simply closed contour *C*, taken in the positive sense and z_0 is any point interior to *C*, then prove that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

(f) State and prove Liouville's theorem.

(g) Suppose that a function f is analytic throughout a disc $|z - z_0| < R_0$ centred at z_0 and with radius R_0. Then prove that $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (|z - z_0| < R_0)$$

where $a_n = \dfrac{f^n(z_0)}{\underline{|n}}$, $(n = 0, 1, 2, \ldots\ldots)$

(h) State and prove Laurent's theorem.

OPTION – D

Paper : MAT–HE–6046

(*Hydromechanics*)

1. Answer the following questions : ***(any ten)*** $1 \times 10 = 10$

   (a) State the property for which a mass of liquid can be used as a 'machine' for the purpose of multiplying power.

   (b) Give example of an elastic fluid.

   (c) Define centre of pressure of a plane area immersed in a fluid.

   (d) Name the instrument used to measure atmospheric pressure.

   (e) Flow elasticity of a fluid is measured ?

   (f) Define absolute zero of temperature.

   (g) What is meant by thermal capacity of a body ?

   (h) Name the *two* methods of treating the general problem of hydrodynamics.

   (i) What is tube of flow ?

   (j) Define a perfect fluid with example.

(k) If $\vec{q}$ be the velocity field of an incompressible homogeneous fluid, what is the value of $\nabla . \vec{q}$?

(l) What is an adiabatic change ?

(m) State Boyle's law.

(n) What is the physical significance of the equation of continuity ?

(o) For what type of motion of a particle, the stream lines and path lines coincidie ?

2. Answer ***any five*** of the following : $2 \times 5 = 10$

(a) Obtain the differential equations of the lines of force at any point.

(b) Prove that the pressure at any point of a fluid varies as the depth of the point from the surface when there is no atmosphere.

(c) "The centre of pressure of any plane area, not horizontal, is below its centroid." Explain why.

(d) Find the equations of the lines of force for a fluid motion in which

$$u = -\frac{c^2 y}{r^2}, \; v = -\frac{c^2 x}{r^2}, \; w = 0$$

where r denotes the distance from z-axis.

(e) What is meant by velocity potential ? Does velocity potential exist for a rotational flow ?

(f) The velocity $\vec{q}$ in the three-dimensional flow field for an incompressible fluid is given by $\vec{q} = \lambda x\hat{i} - y\hat{j} - z\hat{k}$. Find the value of λ.

(g) State the relations among pressure, density and temperature for a given mass of gas under different conditions.

(h) Find the resultant vertical pressure on any surface of a homogeneous liquid at rest under the action of gravity.

3. Answer ***any four*** of the following : 5×4=20

(a) A closed tube in the form of an ellipse with its major axis vertical is filled with three different liquids of densities ρ_1, ρ_2, ρ_3 respectively. If the distances of the surfaces of separation from either focus be r_1, r_2, r_3 respectively, prove that $r_1(\rho_2 - \rho_3) + r_2(\rho_3 - \rho_1) + r_3(\rho_1 - \rho_2) = 0$.

(b) A given volume of liquid is at rest on a fixed plane under the action of a force, to a fixed point in the plane, varying as the distance. Find the pressure at any point of the fluid and the whole pressure on the fixed plane.

(c) A mass of homogeneous liquid, contained in a vessel, revolves uniformly about a vertical axis. Determine the pressure at any point and the surfaces of equal pressure.

(d) A circular area of radius a is immersed with its plain vertical and centre at a depth h. Find the depth of centre of pressure below the free surface.

(e) If the law connecting the pressure and density of the air were $p = k\rho^n$, prove that neglecting variations of gravity and temperature, the height of the atmosphere would be $\dfrac{n}{n-1}$ times the homogeneous atmosphere.

(f) Given the velocity field

$$\vec{q} = Ax^2 y\hat{i} + By^2 z\hat{j} + Czt^2\hat{k}$$

Determine the acceleration of the fluid particle.

(g) A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis. Show that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho_{\omega}) = 0$$

where ω is the angular velocity of a particle whose azimuthal angle is θ at time t.

(h) A vertical rectangle is exposed to the action of the atmosphere at a constant temperature. Assuming that the pressure exerted by the atmosphere varies with the height according to the usual law, determine the position of the centre of pressure of the rectangle.

4. Answer ***any four*** of the following : 10×4=40

(a) If a mass of fluid is at rest under the action of given forces, obtain the equation which determines the pressure at any point of the fluid.

Also deduce a necessary condition of equilibrium. 7+3=10

(b) A liquid of given volume V is at rest under the forces $X = \frac{-\mu x}{a^2}$, $Y = \frac{-\mu y}{b^2}$, $Z = \frac{-\mu z}{c^2}$. Find the pressure at any point. Also find the equation to the free surface and the surfaces of equal pressures.

3

5+2+3=10

(c) A solid octant of a sphere is immersed with one plane face in the surface. Prove that the resultant pressure on the curved surface reduces to a single force of magnitude $\frac{1}{6}\rho\pi a^3\sqrt{\pi^2+8}$

Also show that the equation of the line of action of single resultant is $x = y = \frac{2z}{\pi}$.

8+2=10

(d) Obtain the equation of continuity for a fluid in motion, in the form

$$\frac{\partial \rho}{\partial t} + div\left(\rho\vec{q}\right) = 0$$

6

where ρ and $\vec{q}$ are respectively density and the velocity of the fluid. Deduce the form of equation of continuity when the fluid is homogeneous and incompressible.

8+2=10

Saturn — Buddhiraj das

(e) Define stream lines and path lines. Determine the stream lines and the path of the particles, when the velocity field is given by $u = \dfrac{x}{1+t}$, $v = \dfrac{y}{1+t}$, $w = \dfrac{z}{1+t}$.

4+6=10

(f) Assuming that the temperature of the atmosphere is uniform, show that as the altitude increases in A.P., the pressure decreases in G.P.
Hence deduce a formula for comparing the differences of level of two stations above earth's surface by a barometer.
What changes have to be applied if the temperature is not uniform but acceleration due to gravity is constant?

6+2+2=10

(g) (i) Two volumes v_1 and v_2 of different gases at pressures p_1 and p_2 and absolute temperatures T_1 and T_2 are mixed together, so that the volume of the mixture is V and absolute temperature if T. Prove that the pressure of the mixture is

$$\frac{T}{V}\left(\frac{p_1 v_1}{T_1} + \frac{p_2 v_2}{T_2}\right)$$

(ii) Masses m and m' of two gases in which the ratio of the pressure to the density are respectively k and k', are mixed at the same temperature. Prove that the ratio of the pressure to the density in the compound is $\dfrac{mk+m'k'}{m+m'}$.

5+5=10

(h) Obtain the differential equations of the curves of equal pressure and density for a fluid at rest under the action of the forces x, y, z per unit mass. If $X = y(y+z)$, $Y = z(z+x)$, $Z = y(y-x)$,
5
prove that surfaces of equal pressures are hyperbolic paraboloids $y(x+z) = c(y+z)$ and the curves of equal pressure and density are given by $y(x+z) =$ constant, $y+z =$ constant.

5+5=10

(i) Define the following : 4+6=10

(i) Steady and unsteady flow

(ii) Rotational and irrotational motion

Show that $u = -\dfrac{2xyz}{\left(x^2+y^2\right)^2}$,

$$v = -\frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2}, \quad w = \frac{y}{x^2+y^2}$$

are the velocity components of a possible liquid motion.

(j) A given mass of air is contained within a closed air-tight cylinder with its axis vertical. The air is rotating in relative equilibrium about the axis of the cylinder. The pressure at the highest point of its curved surface is p, and the pressure at the highest point of its axis is P. Prove that, if the fluid were absolutely at rest, the pressure at the upper end of the axis would be $\dfrac{p-P}{\log p - \log P}$, where the weight of the air is taken into account.
