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3 (Sem-5/CBCS) MAT HC 1 (N/O)

2023

MATHEMATICS

(Honours Core)

OPTION-A

(For New Syllabus)

Paper : MAT-HC-5016

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : $1 \times 7 = 7$

(a) Which point on the Riemann sphere represents ∞ of the extended complex plane $\mathbb{C} \cup \{\infty\}$?

(b) A set $S \subseteq \mathbb{C}$ is closed if and only if S contains each of its _____ points.

(Fill in the gap)

Contd.

- (c) Write down the polar form of the Cauchy-Riemann equations.
- (d) The function $f(z) = \sinh z$ is a periodic function with a period, _____ .
(Fill in the gap)
- (e) Define a simple closed curve.
- (f) Write down the value of the integral $\int_C f(z) dz$, where $f(z) = ze^{-2}$ and C is the circle $|z| = 1$.
- (g) Find $\lim_{n \rightarrow \infty} z_n$, where $z_n = -1 + i \frac{(-1)^n}{n^2}$.

2. Answer the following questions : $2 \times 4 = 8$

(a) Let $f(z) = i \frac{z}{2}$, $|z| < 1$. Show that

$$\lim_{z \rightarrow 1} f(z) = \frac{i}{2}, \text{ using } \varepsilon - \delta \text{ definition.}$$

(b) Show that all the zeros of $\sinh z$ in the complex plane lie on the imaginary axis.

(c) Evaluate the contour integral

$$\int_C \frac{dz}{z}, \text{ where } C \text{ is the semi circle}$$
$$z = e^{i\theta}, \quad 0 \leq \theta \leq \pi$$

(d) Using Cauchy's integral formula, evaluate

$$\int_C \frac{e^{2z}}{z^4} dz, \text{ where } C \text{ is the circle } |z| = 1.$$

3. Answer **any three** questions from the following : 5×3=15

(a) Find all the fourth roots of -16 and show that they lie at the vertices of a square inscribed in a circle centered at the origin.

(b) Suppose $f(z) = u(x, y) + iv(x, y)$,
($z = x + iy$) and $z_0 = x_0 + iy_0$,

$w_0 = u_0 + iv_0$. Then prove the following :

$$\lim_{(x, y) \rightarrow (x_0, y_0)} u(x, y) = u_0,$$

$$\lim_{(x, y) \rightarrow (x_0, y_0)} v(x, y) = v_0, \quad \text{if and only}$$

$$\text{if } \lim_{z \rightarrow z_0} f(z) = w_0.$$

(c) (i) Show that the function $f(z) = \operatorname{Re} z$ is nowhere differentiable.

(ii) Let $T(z) = \frac{az + b}{cz + d}$, where

$$ad - bc \neq 0.$$

Show that $\lim_{z \rightarrow \infty} T(z) = \infty$ if $c = 0$.

$$3+2=5$$

(d) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6\pi}{7}$$

(e) State and prove fundamental theorem of algebra.

4. Answer **any three** questions from the following :

$$10 \times 3 = 30$$

(a) (i) Show that $\exp(z + \pi i) = -\exp(z)$

1

(ii) Show that

$$\log(-1+i)^2 \neq 2\log(-1+i)$$

2

(iii) Show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \quad 2$$

(iv) Show that a set $S \subseteq \mathbb{C}$ is unbounded if and only if every neighbourhood of the point at infinity contains at least one point of S . 5

(b) (i) Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$, $g'(z_0)$ exist with $g'(z_0) \neq 0$. Using the definition of derivative show that

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \quad 5$$

(ii) Show that

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n,$$

where $|z| < \infty$. 5

(c) State and prove Laurent's theorem.

(d) (i) Using definition of derivative, show that $f(z) = |z|^2$ is nowhere differentiable except at $z = 0$. 5

- (ii) Define singular points of a function. Determine singular points of the functions :

$$f(z) = \frac{2z+1}{z(z^2+1)} ;$$

$$g(z) = \frac{z^3+i}{z^2-3z+2} \quad 1+4=5$$

- (e) (i) Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D . Prove that the families of curves $u(x, y) = c_1$, $v(x, y) = c_2$ are orthogonal.

- (ii) Let C denote a contour of length L and suppose that a function $f(z)$ is piecewise continuous on C . If M is a non-negative constant such that

$|f(z)| \leq M$ for all z in C
then show that

$$\left| \int_C f(z) dz \right| \leq ML. \quad 5+5=10$$

(f) (i) Prove that two non-zero complex numbers z_1 and z_2 have the same moduli if and only if $z_1 = c_1 c_2$, $z_2 = c_1 \bar{c}_2$, for some complex numbers c_1, c_2 . 4

(ii) Show that mean value theorem of integral calculus of real analysis does not hold for complex valued functions $w(t)$. 3

(iii) State Cauchy-Goursat theorem. 1

(iv) Show that $\lim_{z \rightarrow \infty} \frac{z^2 + 1}{z - 1} = \infty$. 2