J. Assam Sc Soc
Vol. 46, December 2005, PP 44-51

METHOD OF CHARACTERISTICS AND SOLUTION OF DGLAP EVOLUTION EQUATION IN LEADING ORDER AT SMALL-X

R. Baishya[a], R. Rajkhowa[b] And J. K. Sarma[b*]
[a]Physics Department, J.N. College, Boko-781123, Assam, India.
[b]Physics Department, Tezpur University, Napam-784028, Assam, India.
Email : jks@tezu.ernet.in

Received in September 2005 *Revised and accepted in December 2005*

In this paper, the singlet and non-singlet structure functions have been obtained by solving Dokshitzer-Gribove-Lipatov-Altaralli-Parisi (DGLAP) evolution equations in leading order (LO) at the small-x limit. Here we have used a Taylor series expansion and then the method of characteristics to solve the evolution equation. We have also calculated t and x-evolutions of deuteron structure function and the results are compared with the New Moon Collaboration (NMC) data.

Keywords DIS; DGLAP equation; small-x; method of characteristics; structure function.
PACS Nos. 12.35 Eq ; 12.38 t; 12.39-x; 13.60 Hb.

INTRODUCTION

Deep inelastic scattering (DIS)[1-4] process is one of the basic processes for investigating the structure of hadrons. It is well known, all information about the structure of hadrons participating in DIS comes from the hadronic structure functions. According to QCD, at small value of x and at large values of Q^2 hadrons consist predominately of gluons and sea quarks, where x and Q^2 are Bjorken scaling variable and four momentum transfer in a DIS process respectively. The Dokshitzer - Gribov - Lipatov - Altaralli - Parisi (DGLAP) evolution equations[5-8] give t [$= \ln Q^2/\Lambda^2$). Λ is the QCD cut off parameter] and x evolutions of structure functions. Hence the solutions of DGLAP evolution equations give quark and gluon structure functions that produce ultimately proton, neutron and deuteron structure functions.

Among various methods for solving DGLAP evolution equations, in recent years, an approximate method suitable at small-x has been pursued with considerable phenomenological success[9-12]. That method is very simple and mathematically transparent. In that approach, the DGLAP equations are expressed as partial differential equations PDE) in x and t using the Taylor series expansion of some structure functions valid to be at small-x and particular solutions of the equations have been obtained by different arbitrary linear combinations of U and V of the general solution f(U, V) = 0. But one of the limitations of these solutions is that, as the evolution equations are PDE, their ordinary solutions are not unique solution, but a range of solutions, of course the range is very narrow. On the other hand, this limitation can be overcome by using method of characteristics[13-15].

The method of characteristics is an important technique for solving initial value problems of first order PDE. It turns out that if we change co-ordinates from (x, t) to suitable new co-ordinates (S, τ) then the PDE becomes an ordinary differential equation (ODE). Then we can solve ODE by standard method. In figure 1, the co-ordinates S, τ) are considered as the value of S are changed along a vertical curvy line where τ is constant and τ changes along a horizontal curvy line where S is constant. For t-evolution, we consider as S changes along the characteristic curve [x(S), t(S);

$$I_1^{NS}(x,t) = 2\int_x^1 \frac{dw}{1-w}\left[\left(1+w^2\right)F_2^{NS}\left(\frac{x}{w},t\right) - 2F_2^{NS}(x,t)\right], \quad (3c)$$

where $A_f = \frac{4}{33-2N_f}$, N_f being the flavor number.

Let us introduce the variable u = 1-w and note that

$$\frac{x}{w} = \frac{x}{1-u} = x\sum_{k=0}^{\infty} u^k. \quad (4)$$

Since x<w<1, so 0<u<1-x and hence the series 4) is convergent for |u|<1. Now

$$\frac{x}{w} = \frac{x}{1-u} = x\left(1+\frac{1}{1-u}-1\right) = \left(x+\frac{xu}{1-u}\right).$$

So, using Taylor's expansion series we can rewrite

$F_2^S\left(\frac{x}{w},t\right)$ and $G(\frac{x}{w},t)$ as

$$F_2^S\left(\frac{x}{w},t\right) = F_2^S\left(x+\frac{xu}{1-u},t\right)$$

$$= F_2^S(x,t) + \frac{xu}{1-u}\frac{\partial F_2^S(x,t)}{\partial x} + \frac{1}{2}\left(\frac{xu}{1-u}\right)^2\frac{\partial^2 F_2^S(x,t)}{\partial x^2} + \ldots$$

$$\approx F_2^S(x,t) + \frac{xu}{1-u}\frac{\partial F_2^S(x,t)}{\partial x}, \quad (5a)$$

and

$$G(\frac{x}{w},t) \approx G(x,t) + \frac{xu}{1-u}\frac{\partial G(x,t)}{\partial x}. \quad (5b)$$

Since x is small in our region of discussion, the terms containing x^2 and higher powers of x are neglected. Using equation 5a) and 5b) in equation 3a) and 3b) and performing u-integrations we get

$$I_1^S(x,t) = \{-3+2x+x^2\}F_2^S(x,t) + \{x - x^3 - 2x\ln(x)\}\frac{\partial F_2^S(x,t)}{\partial x}, \quad (6a)$$

$$I_2^S(x,t) = \frac{3}{2}N_f\left[\left(\frac{3}{2}-x+x^2-\frac{2}{3}x^3\right)G(x,t) + \left(-\frac{5}{2}x+3x^2-2x^3+\frac{2}{3}x^4-x\ln(x)\right)\frac{\partial G(x,t)}{\partial x}\right] \quad (6b)$$

Putting equations 6a) and 6b) in equation 1) we get

$$\frac{\partial F_2^S(x,t)}{\partial t} - \frac{A_f}{t}\left[A(x)F_2(x,t) + B(x)\frac{\partial F_2^S}{\partial x} + C(x)G(x,t) + D(x)\frac{\partial G(x,t)}{\partial x}\right] = 0 \quad (7)$$

where

$$A(x) = 2x + x^2 + 4\ln(1-x), \quad (8a)$$

$$B(x) = x - x^3 - 2x\ln(x), \quad (8b)$$

$$C(x) = \frac{3}{2}N_f\left(\frac{2}{3}-x+x^2-\frac{2}{3}x^3\right), \quad (8c)$$

$$D(x) = \frac{3}{2}N_f\left[-\frac{5}{3}+3x^2-2x^3-\frac{2}{3}x^4-x\ln(x)\right]. \quad (8d)$$

Since the DGLAP evolution equations of gluon[15] and singlet[12] structure functions in LO are in the same form of derivative with respect to t, so we can assume

$$G\ x, t) = kx)\ F_2^S\ x, t), \quad (9)$$

where k x) is a suitable function of x or may be a constant. We may assume k x) = k, ae^{bx}, cx^d, where k, a, b, c, d are suitable parameters which can be determined by phenomenological analysis. Of course k may be a function of t also.

Now equation (7) gives

$$-t\frac{\partial F_2^S(x,t)}{\partial t} + A_f M\frac{\partial F_2^S(x,t)}{\partial x} + A_f L F_2^S(x,t) = 0 \quad (10)$$

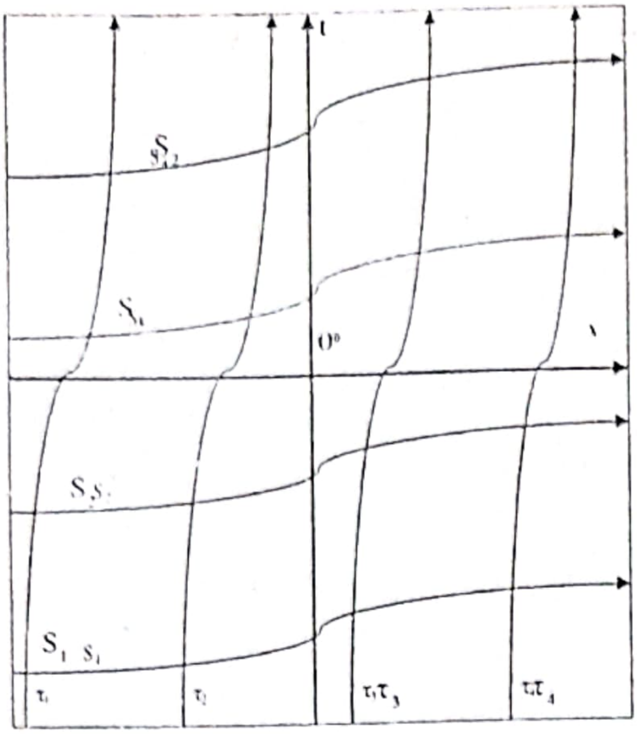

Figure 1. Characteristic curve. For constant values of τ (τ_1, τ_2, τ_3...), the values of S change along a vertical characteristic curve. On the other hand, along a horizontal characteristic curve, the values of τ change for constant values of S (S_1, S_2, S_3 . . .) in the t-x plane.

$0<S<\infty$] and τ changes along the initial ($t = t_0$) curve. On the other hand, for x-evolution, τ changes along the characteristic curve [x (τ), t (τ); $0 < \tau < \infty$] and S changes along the initial curve $x = x_0$). Along the characteristics curve we get one ODE with one boundary condition. After solving it and transforming S, τ) again to x, t) we get unique solution.

In this paper, we obtain a solution of DGLAP equations for singlet and non-singlet structure functions at small-x at LO by using this method of characteristics. The result is compared with NMC data[16] for deuteron structure function. Here the section 1 is the introduction, section 2 deals with the necessary theory and section 3 is the results and discussion.

THEORY

Let us consider a first order linear PDE $a(x,t)U_x + b(x,t)U_y + c(x,t)U = 0$ with initial condition $U(x, 0) = f(x)$. Then the goal of the method of characteristics when applied to this equation, is to change the coordinate system from. (x, t) to a new coordinate system (x_0, S) in which the PDE becomes an ODE along certain curves in the x-t plane. Such curves, along which the solution of the PDE reduces to an ODE, are called characteristic curves or characteristics. The new variable S will vary, and the new variable x_0 will constant along the characteristics. The variable x_0 will change along the initial curve in the x-t plane (along the line t=0). If we consider $\frac{dx}{dS} = a(x,t)$ and $\frac{dt}{dS} = b(x,t)$ then we have

$$\frac{dU}{dS} = \frac{dx}{dS}U_x + \frac{dt}{dS}U_y \qquad = a(x,t)\,U_x + b(x,t)U_y$$

and the PDE becomes the ODE $\frac{dU}{dS} + c(x,t)U = 0$.

which can be easily solved.

Now the DGLAP evolution equations for singlet and non-singlet structure functions in LO in standard form are

$$\frac{\partial F_2^S}{\partial t} - \frac{A_f}{t}\left[\{3 + 4\ln(1-x)\}F_2^S(x,t) + I_1^S(x,t) + I_2^S(x,t)\right] = 0 \tag{1}$$

$$\frac{\partial F_2^S}{\partial t} - \frac{A_f}{t}\left[\{3 + 4\ln(1-x)\}F_2^{NS}(x,t) + I_1^{NS}(x,t)\right] = 0 \tag{2}$$

where

$$I_1^S(x,t) = 2\int_x^1 \frac{dw}{1-w}\left[\left(1+w^2\right)F_2^S\left(\frac{x}{w},t\right) - 2F_2^S(x,t)\right], \tag{3a}$$

$$I_2^S(x,t) = \frac{3}{2}N_f\int_x^1 \left\{w^2 + (1-w)^2\right\}G\left(\frac{x}{w},t\right)dw, \tag{3b}$$

and

$$F_2^S = F_2^S(S_0)\exp\left(-\int_{x=x_0}^{1} \frac{L}{M}dx\right)$$

where L and M are given by equations (11c) and (11d). Now input function can be defined, when $x = x_0$ as $F_2^S(S_0) = F_2^S(x_0, t)$. So

$$F_2^S(x,t) = F_2^S(x_0,t)\exp\left(-\int_{x=x_0}^{1} \frac{L}{M}dx\right) \qquad (20)$$

This is the x- evolution of singlet structure function. Proceeding in the same way, we get t and x evolutions of non-singlet structure function from equation (2) as

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t}{t_0}\right)^{A_f\{2x+x^2+4\ln(1-x)\}} \qquad (21a)$$

and

$$F_2^{NS}(x,t) = F_2^{NS}(x_0,t)\exp\left[-\int_{x=x_0}^{1} \frac{\{2x+x^2+4\ln(1-x)\}}{\{x - x^3 - 2x\ \ln(x)\}}dx\right] \qquad (21b)$$

respectively.

The deuteron, proton and neutron structure functions measured in DIS can be written in terms of singlet and non-singlet quark distribution functions as

$$F_2^d(x,t) = \frac{5}{9}F_2^S(x,t), \qquad (22a)$$

$$F_2^p(x,t) = \frac{5}{18}F_2^S(x,t) + \frac{3}{18}F_2^{NS}(x,t) \qquad (22b)$$

and

$$F_2^n(x,t) = \frac{5}{18}F_2^S(x,t) - \frac{3}{18}F_2^{NS}(x,t). \qquad (22c)$$

The t and x-evolutions of deuteron structure functions can be obtained by putting equations (18) and (20) respectively in the equation (22a) as

$$F_2^d(x,t) = F_2^d(x,t_0)\left(\frac{t}{t_0}\right)^{A_f\left[(2x+x^2+4\ln(1-x)) + \frac{3}{2}N_f k\left(\frac{2}{3} - x + x^2 - \frac{2}{3}x^3\right)\right]} \qquad (23a)$$

and

$$F_2^d(x,t) = F_2^d(x_0,t)\exp\left[-\int_{x=x_0}^{1} \frac{(2x+x^2+4\ln(1-x)) + \frac{3}{2}N_f k\left(\frac{2}{3} - x + x^2 - \frac{2}{3}x^3\right)}{\{x - x^3 - 2x\ln(x)\} + \frac{3}{2}N_f k\left\{-\frac{5}{3}x + 3x^2 - 2x^3 + \frac{2}{3}x^4 - x\ln(x)\right\}}\right]. \qquad (23b)$$

where

$$F_2^d(x,t_0) = \frac{5}{2}F_2^S(x,t_0) \qquad (24a)$$

and

$$F_2^d(x_0,t) = \frac{5}{2}F_2^S(x_0,t). \qquad (24b)$$

RESULTS AND DISCUSSION

In this paper, we compare our result of t and x evolutions of deuteron structure function F_2^d measured by the NMC in muon deuteron DIS with incident momentum 90, 120, 200, 280 GeV[19]. Since the equation (18) and (21a) as well as (20) and (21b) are not in the same form, so we need to separate the input functions from the data points to extract the t and x-evolution of proton where

$$L = A(x) + k(x)C(x) + D(x)\,\partial(kx)/\partial x. \quad (11a)$$

$$M = B(x) + k(x)D(x). \quad (11b)$$

For simplicity let us consider $(k\ x) = k$ (arbitrary constant). Then

$$L = A(x) + k\,C(x)$$

$$= [2x + x^2 + 4\ln(1-x)] + \frac{3}{2}N_f K$$

$$\left(\frac{2}{3} - x + x^2 + -\frac{2}{3}x^3\right). \quad (11c)$$

$$M = \left[x - x^3 - 2x\ln(x)\right] + \frac{3}{2}N_f k\left[-\frac{5}{3}x + 3x^2 - \right.$$

$$\left. 2x^3 + \frac{2}{3}x^4 - x\ln(x)\right]. \quad (11d)$$

Let us introduce the new variable S with characteristics equations

$$\frac{dt}{dS} = -t, \quad (12a)$$

$$\frac{dx}{dS} = A_f M. \quad (12b)$$

Putting equations (12a) and (12b) in equation (10), we get

$$\frac{dF_2^S}{dS} + A_f LF_2^S = 0. \quad (13)$$

Then equation 13) gives

$$\frac{dF_2^S}{F_2^S} = -A_f LdS. \quad (14)$$

Here dS can be defined from either equation (12a) or (12b).

2(a). t - Evolution:

From equations (12a) and (14) we get

$$\frac{dF_2^S}{F_2^S} = -A_f L\left(-\frac{dt}{t}\right). \quad (15a)$$

Integrating we get

$$\ln F_2^S = A_f L\ln(t) + C_1. \quad (15b)$$

where C_1 is the constant of integration. Let us consider, when $S = 0$, $t = t_0$, $F_2^S(S) = F_2^S(0)$.

Then equation 15b) gives

$$C_1 = \ln F_2^S(0) - A_f L\ln(t_0) \quad (16)$$

and

$$F_2^S = F_2^S(0)\left(\frac{t}{t_0}\right)^{A_f L}. \quad (17)$$

Now we have to replace the co-ordinate S to x, t) in equation 17), when $S = 0, t = t_0$; then the input function is $F_2^S(0) = F_2^S(x, t_0)$. So

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t}{t_0}\right)^{A_f\left[\left(2x + x^2 + 4\ln(1-x)\right) + \frac{3}{2}N_f k\left(\frac{2}{3} - x + x^2 - \frac{2}{3}x^3\right)\right]}. \quad (18)$$

This is the t- evolution of singlet structure function.

2(b). x - Evolution :

From equation (12b) and (14), we get

$$\frac{dF_2^S}{F_2^S} = -A_f L\frac{dx}{A_f M}.$$

Integrating the above equation we get

$$\ln F_2^S = -\int \frac{L}{M}dx + C_2. \quad (19a)$$

Let when $x = x_0$ then $F_2^S = F_2^S(S_0)$. So

$$C_2 = \ln F_2^S(S_0) + \int_{x=x_0} \frac{L}{M}dx \quad (19b)$$

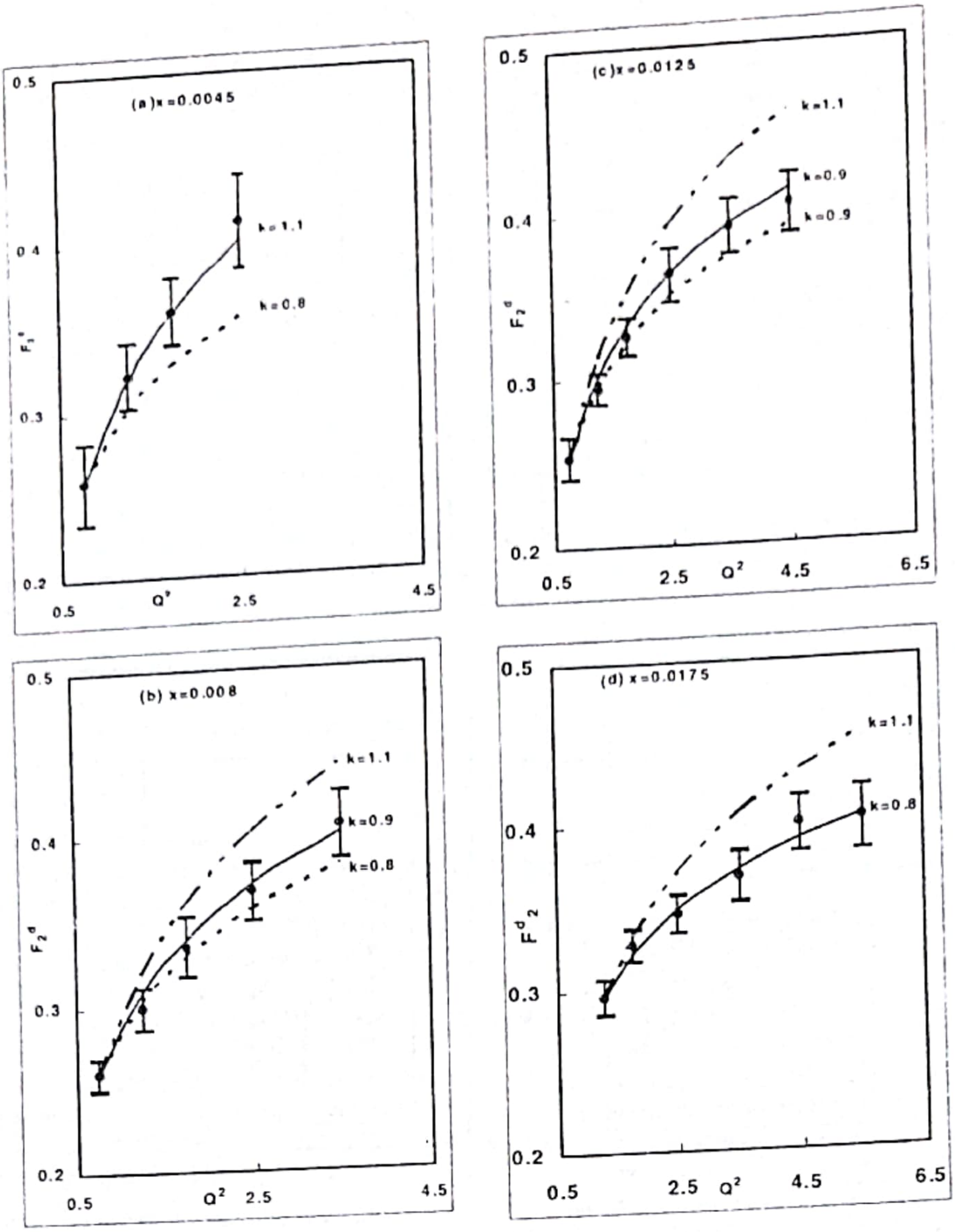

Figure 2. Results of t-evolution of deuteron structure function F_2^d for the given value of x. Data points at lowest-Q^2 values in the figures are taken as input to test the evolution equation (23a).

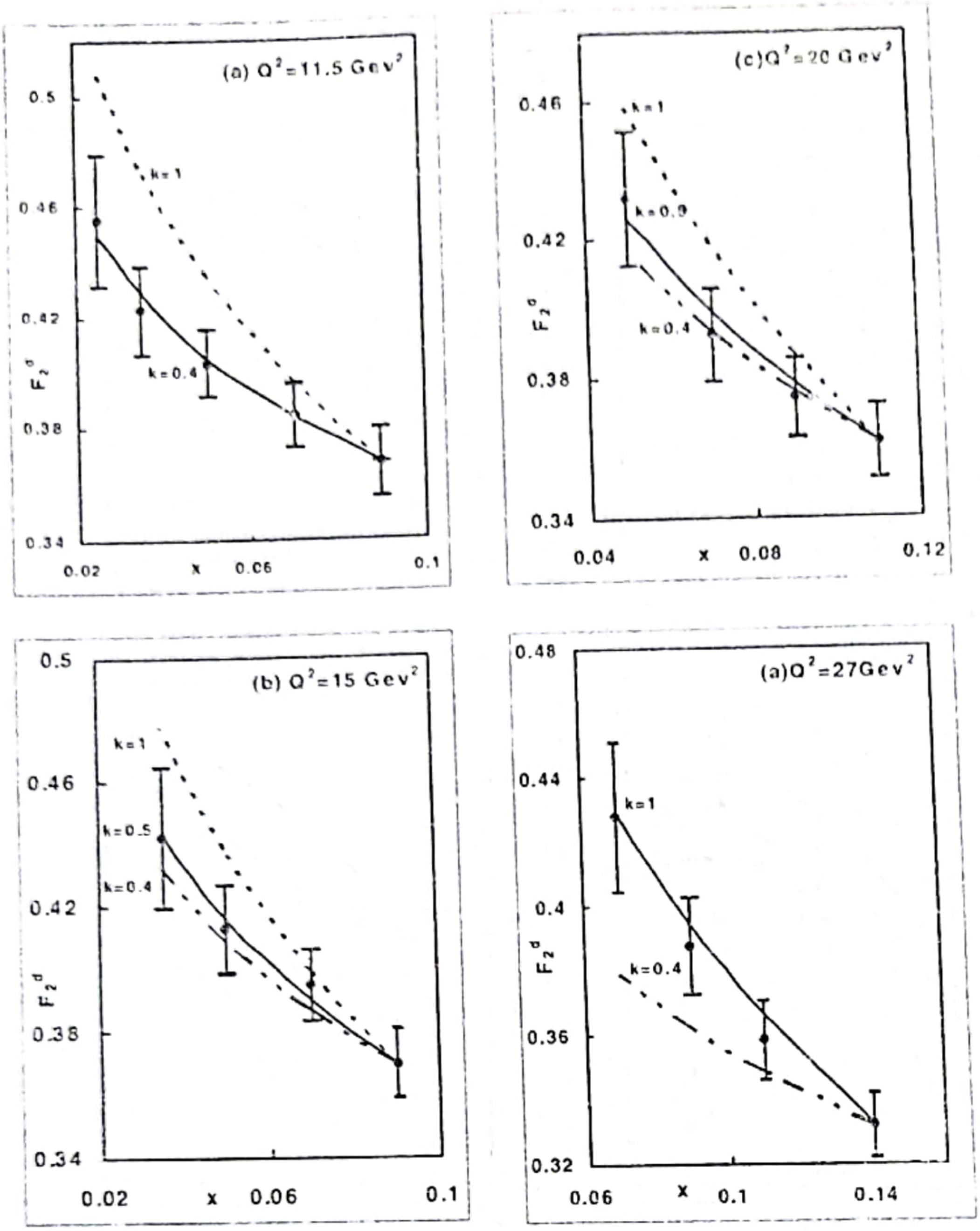

Figure 3. Results of x-evolution of deuteron structure function F_2^d for the given values of t. Data point at lowest-x values in the figures are taken as inputs to test the evolution equation (23b).

and neutron structure function. So using equations (22b) and (22c), evolutions of proton and neutron structure functions are not possible. For quantitative analysis, we consider the QCD cut-off parameter $\Lambda_{\overline{MS}} = 0.323$ GeV [20] for $\alpha_S(M_Z^2) = 0.119 \pm 0.002$ and $N_f = 4$. It is observed that our result is very sensitive to arbitrary constant k in t-evolution and best fitting is in the range of $0.9 = k = 1.1$. In figure 2 for t-evolution, we have plotted computed values of F_2^d against Q^2 values for a fixed x.

In figure (2a), we have plotted the graph for x = 0.0045 with $Q_0^2 = 0.75$ GeV^2 as the initial point. The agreement of our result with the data is excellent at k = 1.1. Similarly in figures (2b), (2c) and (2d) for x = 0.008, x = 0.0125 and x = 0.0175 respectively, the computed values are plotted against the corresponding values of Q^2 for the range 0.75 GeV^2 to 5.5 GeV^2. Here the input parameters are taken as for $Q_0^2 = 0.75$ GeV^2 in first two curves and $Q_0^2 = 1.25$ GeV^2 for the third curve. It is found that agreement of these results with data is excellent for the range $0.8 \le k \le 0.9$. In figure 2, the solid lines represent the best fitcurves. Except the best. fit-curves, the dotted lines represent those for k = 0.8 and dashed lines represent for k = 1.1.

In figure 3 for x-evolution, we have plotted computed values of F_2^d against the x values for a fixed Q^2. Here we have plotted the graphs for Q^2 = 11.5, 15, 20 and 27 GeV^2 for the range of $0.025 = x = 0.14$. Here we have considered the input parameter $x_0 = 0.09$ for first three curves and $x_0 = 0.14$ for the fourth one. The best value of k is k = 0.5. But as Q^2 increases the k value slightly increases. For Q^2 = 27 GeV^2, the excellent agreement is found for k = 1. Here also the solid lines represent the best-fit curves. Except best-fit curves, the dotted lines represent the graphs for k = 1 and dashed line represents for k = 0.4.

Though there are various methods to solve DGLAP evolution equation to calculate quark and gluon structure functions, our method of characteristics to solve these equations is also a reliable alternative. Though mathematically rigorous, it changes the integro-differential equations into ODE and then makes it possible to obtain unique solutions. As our subsequent work we can calculate the t and x - evolution of nucleon structure functions by considering $k(x) = ax^b$ and $k(x) = ce^{dx}$. Also we will try to extend our work to next-to-leading order (NLO) at small-x.

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